[4] R. Kumar, S. Jiang, C. Zhou, and W. Qiu, "Polynomial synthesis of supervisor for partially observed discrete-event systems by allowing nondeterminism in control," *IEEE Trans. Autom. Control*, vol. 50, no. 4, pp. 463–475, Apr. 2005.
[5] S. Lafortune and E. Chen, "The infimal closed controllable superlanguage and its application in supervisory control," *IEEE Trans. Autom. Control*, vol. 35, no. 4, pp. 398–404, Apr. 1990.
[6] F. Lin and W. M. Wonham, "On observability of discrete-event systems," *Inform. Sci.*, vol. 44, no. 3, pp. 173–198, 1988.
[7] R. Milner, *Communication and Concurrency*. New York: Prentice-Hall, 1989.
[8] R. Paige and R. E. Tarjan, "Three partition refinement algorithms," *SIAM J. Comput*, vol. 16, no. 6, pp. 973–989, 1987.
[9] P. J. Ramadge and W. M. Wonham, "Supervisory control of a class of discrete event processes," *SIAM J. Control Optim.*, vol. 25, no. 1, pp. 206–230, 1987.
[10] C. Zhou and R. Kumar, "Bisimilarity control of partially observed deterministic discrete event systems," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1642–1653, Sep. 2007.
[11] C. Zhou and R. Kumar, "Control of nondeterministic discrete event systems for simulation equivalence," *IEEE Trans. Autom. Sci. Eng.*, vol. 4, pp. 340–349, Jul. 2007.
[12] C. Zhou and R. Kumar, "A small model theorem for bisimilarity control under partial observation," *IEEE Trans. Autom. Sci. Eng.*, vol. 4, no. 1, pp. 93–97, Jan. 2007.
[13] C. Zhou, R. Kumar, and S. Jiang, "Control of nondeterministic discrete event systems for bisimulation equivalence," *IEEE Trans. Autom. Control*, vol. 51, no. 5, pp. 754–765, May 2006.

Adaptive Tracking Control of A Class of First-Order Systems With Binary-Valued Observations and Time-Varying Thresholds

Jin Guo, Ji-Feng Zhang, and Yanlong Zhao

Abstract—**This technical note studies the adaptive tracking control for a class of single parameter systems with binary-valued observations and time-varying thresholds. A projection algorithm is proposed for parameter identification, based on which an adaptive control law is designed via the certainty equivalence principle. By use of the conditional expectation of the binary-valued observations with respect to the estimates, it is shown that the identification algorithm is both almost surely and mean square convergent, the closed-loop system is stable, and the adaptive tracking control is asymptotically optimal. A numerical example is given to demonstrate the effectiveness of the algorithms and the main results obtained.**

Index Terms—**Adaptive control, binary-valued observation, optimal tracking, parameter identification, stochastic system.**

I. Introduction

Recently, a class of widely used limited-information systems—set-valued observation systems have attracted a lot of attention ([1]–[6]).

Manuscript received November 10, 2010; March 04, 2011; accepted March 04, 2011. Date of publication July 14, 2011; date of current version December 07, 2011. The work was supported by the National Natural Science Foundation of China under Grants 60934006 and 6657100 and the Special Foundation of President of the Chinese Academy of Sciences under Grant 4581900. Recommended by Associate Editor X. Xia.

The authors are with the Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (e-mail: guojin@amss.ac.cn, jif@iss.ac.cn, ylzhao@amss.ac.cn).

Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2011.2161836

The controlled output of such systems cannot be measured, what can be measured and used for designing controller is whether or not the output belongs to some known set. An epitome of such systems is the binary-valued one ([3], [4]) whose observation space is consisted of two sets, which is tied up with the threshold that may be fixed ([3]) or time-varying ([4]). The observation only tells us the size relationship between the observation and the threshold.

Binary-valued observation system is very common in practical systems due to the widespread use of binary-valued sensors, such as photoelectric sensors for positions, Hall-effect sensors for speed and acceleration, EGO oxygen sensors in automotive emission control, a one-bit quantizer in analog-to-digital conversion, etc. ([3], [4], [7]–[9]). In particular, the theory of binary-valued observation systems with time-varying thresholds can be applied to wireless sensor networks (WSNs) ([5], [6]), which have received much concern due to their potential applications in military surveillance, environmental monitoring, health care, building automation, etc. ([9]). WSNs are generally composed of a large number of low-quality sensors which are equipped with limited computation and communication capabilities and limited energy. Therefore, many researchers are currently engaged in developing energy-efficient algorithms for information processing with the focus on using the quantized messages ([10]–[12]).

Different from the conventional systems, set-valued observation systems tell us very limited information in each measurement. The relationships between the measured signals and the input, state and controlled output are not one-to-one, but essentially nonlinear. Identification and adaptive control methods for conventional linear and nonlinear systems cannot be applied to such systems. New algorithms and methods are needed to be developed for parameter identification, adaptive control and performance analysis of the set-valued observation systems.

Some works ([1]–[3], [5], [6], [8], [13], [14]) have already been done in parameter identification, state estimation and stabilization control. [3] and [13] gave a strongly consistent and asymptotically optimal parameter identification algorithm based on the periodic input and statistical properties of the system noises. [2] proposed a method for designing optimal periodic input to reduce the time complexity on parameter identification. [1] discussed the linear system identification with the colored noises based on multi-sine input signal. [14] studied the identification of quantization systems under a class of deterministic persistent excitation input. Under the Gaussian assumption on the predicted density, [5] and [6] investigated the minimum mean quare filtering using the quantized innovations. [4] and [15] considered the case where the parameters are known, and proposed a state observer and a stabilization control.

In this technical note, we will try to attack the adaptive control problem of set-valued observation systems. By using parameter estimates and control inputs to adjust the thresholds, a projection algorithm is proposed to estimate the unknown parameters, and by certainty equivalence principle, an adaptive tracking control is constructively designed. Under some mild *a priori* information on the unknown parameters, statistical properties of the noises and the signals to be tracked, it is shown that the identification algorithm is both almost surely and mean square convergent, the closed-loop system is stable, and, the adaptive tracking control is asymptotically optimal.

For the binary-valued observation systems, compared to the parameter identification and the stabilization control when the parameters are known, the adaptive control is much more difficulty. One main reason is the inter-dependence between the adaptive control law and parameter estimates. In existing works, the property of the identification algorithm is guaranteed by assuming that the input is periodic and deterministic.

However, in adaptive control, the input is required to be adjusted according to the control objectives and parameter estimates, as a result, is neither periodic nor deterministic. Another main reason is lacking of efficient method to deal with the strong nonlinearity and complexity of the stochastic processes resulted from the set-valued observations, the estimation algorithm and the feedback.

To overcome these difficulties, we get rid of the restrictions of the periodicity ([3]) and determinacy ([14]) on the inputs, remove the Gaussian assumption on the predicted density ([5], [6]), and make it feasible to develop a control-dependent online identification algorithm. For the general set-valued observation systems, such kind of extension is very difficult. Therefore, as the first step, we will only consider a class of single parameter systems with binary-valued observation in this technical note.

This technical note is organized as follows. Section II formulates the problem. Section III gives a projection algorithm for parameter identification and a constructive method of designing adaptive control. Section IV analyzes the performance of the closed-loop system, including the convergence rate of the identification algorithm, the stability of the closed-loop systems and the optimality of the adaptive control. Section V uses a numerical example to demonstrate the effectiveness of the algorithms and the main results obtained. Section VI gives some concluding remarks.

II. Problem Formulation

Consider the following first-order system:

$$\begin{cases} y_k = \theta u_k + d_k \\ q_k = \mathcal{Q}(y_k) \end{cases} \tag{1}$$

where $u_k \in \mathbb{R}$, $\theta \in \mathbb{R}$ and $\{d_k, k \geq 1\}$ are, respectively, the input, unknown parameter and noise; $y_k \in \mathbb{R}$ is the controlled output, which cannot be exactly measured but is the target signal to be regulated. y_k is measured by a sensor whose output q_k is binary-valued with threshold c_k, which is a design variable in this technical note. The sensor can be represented by

$$q_k = \mathcal{Q}(y_k) = I_{[y_k > c_k]} - I_{[y_k \leq c_k]} = \begin{cases} 1, & \text{if } y_k > c_k; \\ -1, & \text{otherwise.} \end{cases} \tag{2}$$

The purpose of this technical note is to design an adaptive control to drive the controlled output y_k to follow a known reference signal $\{y_k^*\}$. In other words, at time instant k, we will construct an adaptive control u_k based on the past observations $\{q_1, \dots, q_{k-1}, u_1, \dots, u_{k-1}\}$ to minimize the following tracking index:

$$J_k = E\left(y_k - y_k^*\right)^2. \tag{3}$$

To do so, we need to consider the choice of the threshold c_k, the identification of the unknown parameter and the design of the control law. Comparing with the conventional adaptive control ([16]), we not only have to design the thresholds c_k, but also face the difficulty that the observation is nonlinear and provides very limited information.

For convenience of citation, we now list the main conditions to be used in this technical note:

Assumption 1: The prior information of θ is that $|\theta| \in [\underline{\theta}, \overline{\theta}]$, where $\overline{\theta}$ and $\underline{\theta}$ are known constants with $\overline{\theta} > \underline{\theta} > 0$.

Assumption 2: $\{d_k, k \geq 1\}$ is an independent and identically distributed (i.i.d.) stochastic sequence and d_1 is a normally distributed random variable with zero mean and known covariance σ^2 denoted by $d_1 \sim N(0, \sigma^2)$.

Assumption 3: The target output $\{y_k^*, k \geq 1\}$ is a deterministic signal sequence, and there are known constants $\underline{y}^*$ and $\bar{y}^*$ with $\bar{y}^* \geq \underline{y}^* > 0$ such that $|y_k^*| \in [\underline{y}^*, \bar{y}^*]$.

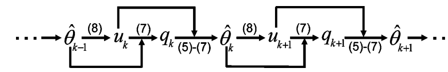

Fig. 1. Procedure of designing the adaptive control law.

Remark 1: Assumption 1 not only implies that the system is controllable, but also tells us the controllability degree of the system (1). Assumption 3 describes the properties of the reference signals, based on which a control law can be designed to ensure a sufficient persistent excitation condition for parameter estimation.

III. Control Law

We first consider the case where the parameter θ is known. In this case, the control law that minimizes (3) should satisfy

$$y_k^* = \theta u_k. \tag{4}$$

Substituting (4) into (1), we obtain $y_k - y_k^* - d_k = 0$, and then $J_k = E(y_k - y_k^*)^2 = Ed_k^2 = Ed_1^2$. However, in the case where the parameter θ is unknown, we need to estimate it. To do so, we propose the following recursive projection algorithm:

$$\hat{\theta}_k = \Pi_\Theta \left\{ \hat{\theta}_{k-1} + \beta_1 \frac{P_{k-1} u_k}{1 + P_{k-1} u_k^2} q_k \right\} \tag{5}$$

$$P_k = P_{k-1} - \beta_2 \frac{P_{k-1}^2 u_k^2}{1 + P_{k-1} u_k^2} \tag{6}$$

$$q_k = I_{[y_k > \hat{\theta}_{k-1} u_k]} - I_{[y_k \leq \hat{\theta}_{k-1} u_k]} \tag{7}$$

where $\Theta \triangleq [-\overline{\theta}, \overline{\theta}]$, initial value $|\hat{\theta}_0| \in [\underline{\theta}, \overline{\theta}]$ and $P_0 > 0$ can be arbitrarily chosen, $\beta_1 > 0$ and $\beta_2 \in (0, 1]$ are two real numbers. $\Pi_\Theta(\cdot)$ is the projection operator, i.e., $\Pi_\Theta(x) = \operatorname{argmin}_{z \in \Theta} |x - z|$, for any $x \in \mathbb{R}$.

According to the certainty equivalence principle, replacing the θ in (4) by its estimate $\hat{\theta}_{k-1}$, we obtain the adaptive control law $y_k^* = \hat{\theta}_{k-1} u_k$. Since u_k cannot be well defined when $\hat{\theta}_{k-1} = 0$, we make the following modification:

$$u_k = \frac{y_k^*}{\hat{\theta}_{k-1}} I_{[\underline{\theta} \leq |\hat{\theta}_{k-1}| \leq \overline{\theta}]} + \frac{y_k^*}{\underline{\theta}} \left(I_{[0 < \hat{\theta}_{k-1} < \underline{\theta}]} - I_{[-\underline{\theta} < \hat{\theta}_{k-1} \leq 0]} \right). \tag{8}$$

From Section IV, it can be seen that β_1 and β_2 in (5)–(6) and their ratio β_1/β_2 have important influence on the the convergence rate of the parameter estimation error.

Remark 2: The design procedure of the adaptive control is shown in Fig. 1. Firstly, calculate $\hat{\theta}_{k-1}$ by use of q_{k-1} according to (5)–(7), and u_k according to (8). Secondly, use u_k to control the system and the sensor to measure the sign of $y_k - \hat{\theta}_{k-1} u_k$, i.e., q_k. Then, repeat the process.

IV. Performance of the Closed-Loop System

Before analyzing the stability of the closed-loop system and the optimality of the control law (8), we first analyze the convergence of the identification algorithm (5)–(7).

A. Convergence of the Identification Algorithm

Denote the estimation error $\tilde{\theta}_k \triangleq \hat{\theta}_k - \theta$. Notice that Θ is a convex-compact set, by the property of the projection operator, we have

$$|\tilde{\theta}_k| \leq \left| \tilde{\theta}_{k-1} + \beta_1 \frac{P_{k-1} u_k}{1 + P_{k-1} u_k^2} q_k \right| \tag{9}$$

and by (1) and (7), q_k can be rewritten in the following form:

$$q_k = I_{[d_k > u_k \tilde{\theta}_{k-1}]} - I_{[d_k \leq u_k \tilde{\theta}_{k-1}]}. \tag{10}$$

Theorem 1: For the system (1), under the conditions of Assumptions 1 and 2, if $u_k \in \mathscr{F}_{k-1} = \sigma(d_i, 1 \leq i \leq k-1)$, and there exist constants $M_2 > M_1 > 0$ such that

$$M_1 \leq |u_k| \leq M_2 \tag{11}$$

then the parameter estimation error $\tilde{\theta}_k$ given by the algorithm (5)–(7) have the following properties:

$$\lim_{k\to\infty} E\tilde{\theta}_k^2 = 0, \quad \lim_{k\to\infty} \tilde{\theta}_k = 0 \text{ a.s.} \tag{12}$$

Proof: By (9) we have

$$\tilde{\theta}_k^2 \leq \tilde{\theta}_{k-1}^2 + \frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2} + 2\frac{\beta_1 P_{k-1} u_k}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1} q_k$$

which together with $u_k \in \mathscr{F}_{k-1}$ renders

$$\begin{aligned} E[\tilde{\theta}_k^2 | \mathscr{F}_{k-1}] &\leq \tilde{\theta}_{k-1}^2 + \frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2} \\ &\quad + 2\frac{\beta_1 P_{k-1} u_k}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1} E[q_k|\mathscr{F}_{k-1}] \\ &= \tilde{\theta}_{k-1}^2 + \frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2} \\ &\quad + 2\frac{\beta_1 P_{k-1} u_k}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1} \\ &\quad \times \left(1 - 2\Phi(u_k\tilde{\theta}_{k-1})\right), \end{aligned} \tag{13}$$

where $\Phi(x)$ is the distribution function of d_1, i.e., $\Phi(x) = 1/\sqrt{2\pi}\sigma \int_{-\infty}^{x} e^{-u^2/2\sigma^2} du$.

By Assumption 1 and (5), we know that

$$|\tilde{\theta}_k| \leq 2\bar{\theta}. \tag{14}$$

Let $\alpha = 2\bar{\theta}M_2$. Then, by Lemma 1, there exists $\bar{B}_1 = \bar{B}_1(\alpha)$ such that $u_k\tilde{\theta}_{k-1}\left(1 - 2\Phi(u_k\tilde{\theta}_{k-1})\right) \leq -\bar{B}_1 u_k^2 \tilde{\theta}_{k-1}^2$. Substituting this into (13) results in

$$\begin{aligned} &E[\tilde{\theta}_k^2|\mathscr{F}_{k-1}] \\ &\leq \tilde{\theta}_{k-1}^2 + \frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2} - 2\beta_1\bar{B}_1 \frac{P_{k-1}u_k^2}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1}^2 \\ &= \left(1 - 2\beta_1\bar{B}_1\frac{u_k^2}{u_k^2 + P_{k-1}^{-1}}\right)\tilde{\theta}_{k-1}^2 + \frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2}. \end{aligned}$$

Since $M_1 \leq |u_k| \leq M_2$ and (20), we have

$$\begin{aligned} &E[\tilde{\theta}_k^2|\mathscr{F}_{k-1}] \\ &\leq \left(1 - \frac{2\beta_1\bar{B}_1 M_1^2}{M_2^2 + P_0^{-1} + \beta_2 M_2^2(k-1)}\right)\tilde{\theta}_{k-1}^2 \\ &\quad + \beta_1^2 M_2^2\left(\frac{1}{P_0} + \frac{\beta_2 M_1^2}{(1-\beta_2)P_0M_2^2+1}(k-1)\right)^{-2}. \end{aligned}$$

Therefore

$$\begin{aligned} E\tilde{\theta}_k^2 \leq &\left(1 - \frac{2\beta_1\bar{B}_1 M_1^2}{M_2^2 + P_0^{-1} + \beta_2 M_2^2(k-1)}\right)E\tilde{\theta}_{k-1}^2 \\ &+\beta_1^2 M_2^2\left(\frac{1}{P_0} + \frac{\beta_2 M_1^2(k-1)}{(1-\beta_2)P_0M_2^2+1}\right)^{-2}. \end{aligned}$$

This together with

$$\frac{\beta_1^2 M_2^2\left(\frac{1}{P_0} + \frac{\beta_2 M_1^2(k-1)}{(1-\beta_2)P_0M_2^2+1}\right)^{-2}}{\frac{2\beta_1\bar{B}_1M_1^2}{M_2^2+P_0^{-1}+\beta_2M_2^2(k-1)}} \to 0$$

and [17, Theorem 1.2.22] implies $E\tilde{\theta}_k^2 \to 0$.

Noticing $x(1-2\Phi(x)) = -2x\int_0^x (1/\sqrt{2\pi\sigma})e^{-u^2/2\sigma^2}du \leq 0$, $\forall x \in \mathbb{R}$, we have

$$E[\tilde{\theta}_k^2|\mathscr{F}_{k-1}] \leq \tilde{\theta}_{k-1}^2 + \frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2}$$

and from (20)

$$\begin{aligned} &E\left(\sum_{k=1}^{\infty}\frac{\beta_1^2 P_{k-1}^2 u_k^2}{1+P_{k-1}u_k^2}\right) \\ &\quad < \beta_1^2 M_2^2 \sum_{k=1}^{\infty}\left(\frac{1}{P_0} + \frac{\beta_2 M_1^2(k-1)}{(1-\beta_2)P_0M_2^2+1}\right)^{-2} < \infty. \end{aligned}$$

Thus, by [18, Lemma 1.2.2], we have that $\tilde{\theta}_k$ converges almost surely to a bounded limit. Notice that $E\tilde{\theta}_k^2 \to 0$. Then, there is a subsequence of $\tilde{\theta}_k$ that converges almost surely to 0. Consequently, $\tilde{\theta}_k$ almost surely converges to 0. □

Remark 3: Compared with the previous work, the conditions (11) are essentially different. In the existing literature, the analysis of estimation errors needs the periodicity of inputs ([3]). In fact, Theorem 1 not only removes the requirement for periodic input, but also implements the parameter identification based on feedback control. Furthermore, the adaptive control for the limited information systems becomes possible. This is because, for the adaptive control (8), if $\underline{y}^* \leq |y_k^*| \leq \bar{y}^*$ and $M_1 = \underline{y}^*/\bar{\theta}$, $M_2 = \bar{y}^*/\underline{\theta}$, then conditions (11) is established.

Theorem 2: Under the conditions of Theorem 1, if $2B\beta > (M_2/M_1)^2$, then $E\tilde{\theta}_k^2 = O(k^{-1})$, where $\beta = \beta_1/\beta_2$ and $B = 2\Phi'(0) = 2/\sqrt{2\pi}\sigma$.

Proof: Since $2B\beta > (M_2/M_1)^2$, by Lemma 6, there exists m such that $E\tilde{\theta}_k^{2m} = o\left(k^{-1}\right)$. Furthermore, by Lemma 5, we have $E\tilde{\theta}_k^{2r} = o\left(k^{-1}\right)$, $r = 2, \ldots, m$. Notice that

$$\begin{aligned} \frac{P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2} &\leq \frac{M_2^2}{\left(u_k^2 + P_{k-1}^{-1}\right)^2} \\ &\leq \frac{M_2^2}{\left(M_1^2 + P_0^{-1} + \frac{\beta_2 M_1^2(k-1)}{(1-\beta_2)P_0M_2^2+1}\right)^2} \\ &= O(k^{-2}) \end{aligned}$$

and

$$\frac{P_{k-1}|u_k|}{1+P_{k-1}u_k^2} \leq \frac{M_2}{M_1^2 + P_0^{-1} + \frac{\beta_2 M_1^2(k-1)}{(1-\beta_2)P_0M_2^2+1}} = O(k^{-1}).$$

Let $\alpha = 2\bar{\theta}M_2$. Then, by Lemma 1 we have

$$\begin{aligned} E\tilde{\theta}_k^2 \leq &E\tilde{\theta}_{k-1}^2 \\ &+ E\left[\frac{\beta_1^2 P_{k-1}^2 u_k^2}{(1+P_{k-1}u_k^2)^2}\right. \\ &\qquad \left. + 2\frac{\beta_1 P_{k-1}u_k}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1}\left(1 - 2\Phi(u_k\tilde{\theta}_{k-1})\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\le E\tilde{\theta}_{k-1}^2 - E\left(\frac{2\beta_1 BP_{k-1}u_k^2}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1}^2\right)\\
&\quad + E\left[\frac{P_{k-1}^2u_k^2\beta_1^2}{(1+P_{k-1}u_k^2)^2}\right]\\
&\quad + E\left[\frac{2P_{k-1}u_k\beta_1}{1+P_{k-1}u_k^2}\right.\\
&\qquad\left.\times\left(2\sum_{j=2}^{m-1}\frac{\Phi^{(2j-1)}(0)}{(2j-1)!}\tilde{\theta}_{k-1}^{2j}+\bar{B}_m\tilde{\theta}_{k-1}^{2m}\right)\right]\\
&= E\tilde{\theta}_{k-1}^2 - E\left(\frac{2\beta_1 BP_{k-1}u_k^2}{1+P_{k-1}u_k^2}\tilde{\theta}_{k-1}^2\right) + O\left(\frac{1}{k^2}\right)\\
&\le \left(1-\frac{2\beta B\left(\frac{M_1}{M_2}\right)^2}{k-1+\frac{M_2^2+P_0^{-1}}{\beta_2 M_2^2}}\right)E\tilde{\theta}_{k-1}^2\\
&\quad + O\left(\frac{1}{k^2}\right)
\end{aligned}
$$

and hence, by Corollary 1, we get the theorem. □

Remark 4: Theorem 2 gives us the convergence rate of the algorithm (5)–(7), most importantly, describes the influence of β_1 and β_2 on the performance of the algorithm. This implies that β_1 and β_2 can be chosen such that the convergence rate of the identification algorithm (5)–(7) is of order $1/k$, which is the same as the convergence rate of the minimum mean square error identification algorithm for the conventional system.

B. Stability and Optimality of the Closed-Loop System

Theorem 3: Consider the system (1) under the adaptive control (5)–(8). If Assumptions 1–3 hold, then we have that

(i) the parameter estimates are strongly consistent and mean square convergent to the real parameter: $\lim_{k\to\infty}\tilde{\theta}_k^2 = 0$ a.s., $\lim_{k\to\infty}E\tilde{\theta}_k^2 = 0$;

(ii) the estimates have the following convergence rate $E\tilde{\theta}_k^2 = O\left(k^{-1}\right)$ in the case of $2B\beta > \left[(\bar{y}^*\bar{\theta})/(\underline{y}^*\underline{\theta})\right]^2$.

Proof: Noticing that $u_k \in \mathscr{F}_{k-1}$ and

$$0 < \frac{\underline{y}^*}{\bar{\theta}} \le |u_k| \le \frac{\bar{y}^*}{\underline{\theta}} < \infty \tag{15}$$

by Theorems 1–2, one can get the theorem. □

Theorem 4: Consider the system (1) under the adaptive control (5)–(8). If Assumptions 1–3 hold, then we have that

(i) the closed-loop system is stable: $\sup_{k\ge 0}Ey_k^2 < \infty$;

(ii) the closed-loop system is asymptotically optimal: $\lim_{k\to\infty}J_k = Ed_1^2$.

Proof: By (1) and (8) we have $|y_k| < |d_k| + \bar{y}^*\bar{\theta}/\underline{\theta}$. This together with Assumption 3 implies (i).

By Theorem 3 we have $\hat{\theta}_{k-1}\to\theta$ a.s.; and by (8), $\hat{\theta}_{k-1}u_k - y_k^* \to 0$ a.s.. Thus, $\theta u_k - y_k^* = (\hat{\theta}_{k-1}u_k - y_k^*) - (\hat{\theta}_{k-1}-\theta)u_k \to 0$ a.s.. Furthermore, by Assumptions 3, (15) and the dominated convergence theorem ([19, pp. 100]), we know that

$$E(\theta u_k - y_k^*)^2 \to 0. \tag{16}$$

By Assumptions 2 and (8), d_k and u_k are independent. Thus, $E(y_k - y_k^*)^2 = E(\theta u_k + d_k - y_k^*)^2 = Ed_k^2 + E(\theta u_k - y_k^*)^2 = Ed_1^2 + E(\theta u_k - y_k^*)^2$. This together with (16) renders $J_k = E(y_k - y_k^*)^2 \to Ed_1^2$. Hence, (ii) is true. □

V. Simulation

Consider the following system: $y_k = \theta u_k + d_k$, the quantized output information is $q_k = I_{[y_k > \hat{\theta}_{k-1}u_k]} - I_{[y_k \le \hat{\theta}_{k-1}u_k]}$, the constant parameter $\theta = 3$ is unknown, but its range $[1, 10]$ is known, i.e., $\underline{\theta} = 1, \bar{\theta} = 10$. The system noise $\{d_k, k \ge 1\}$ satisfies Assumption 2 with $\sigma = 0.1$.

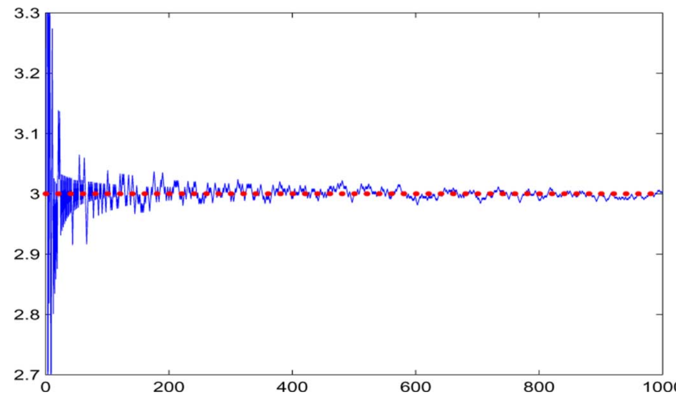

Fig. 2. Convergence of $\hat{\theta}_k$ (solid) to the real parameter $\theta = 3$ (dashed).

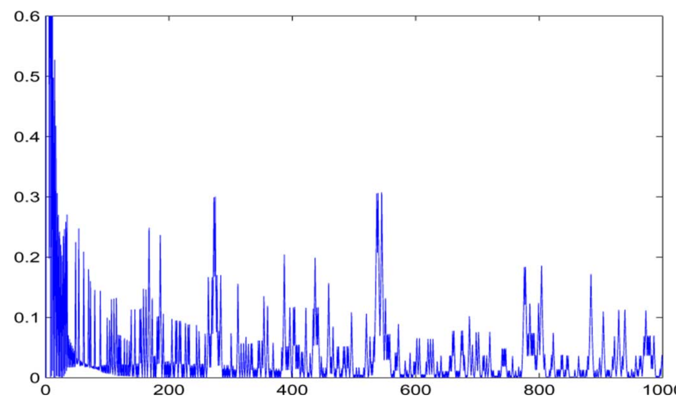

Fig. 3. Sample trajectory of $k(\hat{\theta}_k - \theta)^2$.

Our control purpose is to track the signal $y_k^* \equiv 15$.

Design the following control law:

$$u_k = \frac{15}{\hat{\theta}_{k-1}}I_{[1\le|\hat{\theta}_{k-1}|\le 10]} + \frac{15}{1}\left(I_{[0<\hat{\theta}_{k-1}<1]} - I_{[-1<\hat{\theta}_{k-1}\le 0]}\right) \tag{17}$$

where $\hat{\theta}_k$ is given by (5)–(7) with initial value $\hat{\theta}_0 = P_0 = 1$. Let $\beta_1 = 10$ and $\beta_2 = 0.1$. Then, we have $2B\beta > \left[(\bar{y}^*\bar{\theta})/(\underline{y}^*\underline{\theta})\right]^2$. Thus, according to Theorem 3, we know that

$$\hat{\theta}_k \xrightarrow{a.s.} \theta = 3, E(\hat{\theta}_k - \theta)^2 = O\left(\frac{1}{k}\right), E(y_k - y_k^*)^2 \to Ed_1^2.$$

Let us look at the effectiveness of parameter estimates and tracking from a trajectory.

i) Convergence of the parameter estimates
Fig. 2 shows the convergence of parameter estimates within 1000 steps. Though there is a larger estimation error at the beginning, the estimates eventually converge to the true value.

ii) Convergence rate of parameter estimates
Fig. 3 describes a trajectory of $k(\hat{\theta}_k - \theta)^2$. We can see that this trajectory is bounded, and hence, $(\hat{\theta}_k - \theta)^2 = O(1/k)$.

iii) Tracking performance of the closed-loop system
Fig. 4 shows a trajectory of y_k. We can see that the system output fluctuates around $y_k^* \equiv 15$ under the control (17). The fluctuation is due to the system noise.

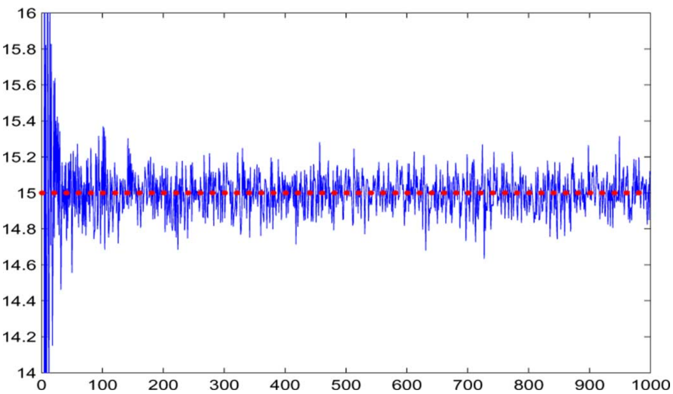

Fig. 4. Tacking performance of y_k (solid) to $y^* = 15$ (dashed).

VI. Conclusion

In this technical note, taking a class of simple linear systems as an example, we made a first step towards the quantitative study of the adaptive control of set-valued observation systems. We proposed a projection algorithms to estimate the unknown parameter, used the parameter estimates and control inputs to adjust the observation thresholds and designed the control law. Under some mild conditions on the *a priori* knowledge of the unknown parameters, statistical properties of the system noises and reference signals, we proved the stability of the closed-loop system and the asymptotical optimality of adaptive tracking, and obtained the convergence rate of identification algorithms. By a numerical example, we also demonstrated the efficiency of the algorithms. From this work, we can imagine that the study on the adaptive control of general set-valued observation systems is more challenging.

Appendix

Lemma 1: Assume that α is a positive real number. Then, for any given positive integer m, there exists $\bar{B}_m = \bar{B}_m(\alpha)$ such that the following inequality holds for all $x \in [-\alpha, \alpha]$:

$$2\sum_{j=1}^{m-1} \frac{\Phi^{(2j-1)}(0)}{(2j-1)!} \cdot x^{2j} + \bar{B}_m x^{2m} \le x(2\Phi(x)-1) \le Bx^2$$

especially, when $m = 1$, we can take $\bar{B}_m = \bar{B}_1 = 1/\alpha$, where $\Phi(x) = 1/\sqrt{2\pi}\sigma \int_{-\infty}^{x} e^{-u^2/2\sigma^2} du$ and $B = 2\Phi'(0) = 2/\sqrt{2\pi}\sigma$.

Proof: Since the functions involved in the inequality are all even, we need only to consider the case of $x \in [0, \alpha]$.

We fist prove the second inequality. Let $y_1(x) = Bx - (2\Phi(x)-1)$. Then $y_1'(x) = B - 2\Phi'(x) = B - Be^{-x^2/2\sigma^2} \ge 0, \forall x \in [0, \alpha]$, and hence, $y_1(x) \ge y_1(0) = 0$, or equivalently, $x(2\Phi(x)-1) \le Bx^2$, $\forall x \in [0, \alpha]$.

Next we prove the first inequality. Let $y_2(x) = 2\Phi(x) - 1 - x/\alpha$. Then $y''_2(x) = -B\sigma^{-2} x e^{-x^2/2\sigma^2} \le 0, \forall x \in [0, \alpha]$. Noticing $y_2(0) = y_2(\alpha) = 0$, we have $x/\alpha \le 2\Phi(x) - 1, \forall x \in [0, \alpha]$. Thus, when $m = 1$, the first inequality is true.

For $m > 1$, we do the Taylor's expansion for $x(2\Phi(x)-1)$ on $[0, \alpha]$:

$$\begin{aligned} x(2\Phi(x)-1) &= 2 \cdot \sum_{j=1}^{m-1} \frac{\Phi^{(2j-1)}(0)}{(2j-1)!} \cdot x^{2j} \\ &+ \frac{2\Phi^{(2m-1)}(0)}{(2m-1)!} x^{2m} + \frac{2\Phi^{(2m)}(\xi x)}{(2m)!} x^{2m+1}, \quad \xi \in (0,1). \end{aligned}$$

Let $\bar{B}_m^0 = \min_{x\in[0,\alpha]} \left\{\Phi^{(2m)}(\xi x) x\right\}$. Then, $\bar{B}_m = \left\{2\Phi^{(2m-1)}(0)/(2m-1)! + 2\bar{B}_m^0/(2m)!\right\}$ can make the inequality hold. □

Lemma 2: For any given $a \in \{x : x \in \mathbb{R}, x \ne -1, -2, \ldots\}$ and $\lambda \in \mathbb{R}$, we have

$$\prod_{j=1}^{k} \left(1 - \frac{\lambda}{j+a}\right) = O\left(\frac{1}{k^\lambda}\right), \quad k \to \infty.$$

Proof: From

$$\begin{aligned} \prod_{j=1}^{k} \left(1 - \frac{\lambda}{j+a}\right) &= \exp\left\{\sum_{j=1}^{k} \log\left(1 - \frac{\lambda}{j+a}\right)\right\} \\ &= O\left(\exp\left\{-\sum_{j=1}^{k} \frac{\lambda}{j+a}\right\}\right) \\ &= O(\exp\{-\lambda \cdot \log k\}) = O\left(\frac{1}{k^\lambda}\right) \end{aligned}$$

we have the lemma. □

Lemma 3: For any $a \in \{x : x \in \mathbb{R}, x \ne -1, -2, \ldots\}$ and $\delta \ge 0$, we have

$$\sum_{j=1}^{k} \prod_{l=j+1}^{k} \left(1 - \frac{\lambda}{l+a}\right) \frac{1}{j^{2+\delta}} = \begin{cases} O\left(\frac{\log k}{k^{1+\delta}}\right), & \lambda = 1+\delta; \\ O\left(\frac{1}{k^{1+\delta}}\right), & \lambda > 1+\delta; \\ O\left(\frac{1}{k^\lambda}\right), & \text{else.} \end{cases} \quad (18)$$

Proof: From

$$\frac{\sum_{j=1}^{k} \prod_{l=j+1}^{k} \left(1 - \frac{\lambda}{l+a}\right) \cdot \frac{1}{j^{2+\delta}}}{\frac{1}{k^\lambda} \sum_{j=1}^{k} \frac{j^\lambda}{j^{2+\delta}}} \to 1$$

one can easily get (18). □

Corollary 1: Suppose that $\{x_k, k \ge 1\}$ is a sequence of real numbers such that for all sufficiently large n

$$x_k \le \left(1 - \frac{\lambda}{k+a}\right) x_{k-1} + \frac{\mu}{(k-1)^{2+\delta}}$$

where $a \in \{x : x \in \mathbb{R}, x \ne -1, -2, \ldots\}, \lambda > 0, \delta \ge 0$. Then

$$x_k = \begin{cases} O\left(\frac{1}{k^\lambda}\right), & 0 < \lambda < 1+\delta; \\ O\left(\frac{\log k}{k^{1+\delta}}\right), & \lambda = 1+\delta; \\ O\left(\frac{1}{k^{1+\delta}}\right), & \lambda > 1+\delta. \end{cases}$$

Corollary 2: Suppose that $\{x_k, k \ge 1\}$ is a sequence of real numbers such that for all sufficiently large n

$$x_k \le \left(1 - \frac{\lambda}{k+a}\right) x_{k-1} + o\left(\frac{1}{(k-1)^{2+\delta}}\right)$$

where $a \in \{x : x \in \mathbb{R}, x \ne -1, -2, \ldots\}, \lambda > 1+\delta, \delta \ge 0$. Then $x_k = o\left(k^{-(1+\delta)}\right)$.

Lemma 4: If there exists constants $M_2 > M_1 > 0$ such that (11) holds, then P_k have the following properties:

i) P_k^{-1} has a recursive form:

$$P_k^{-1} = P_{k-1}^{-1} + \frac{\beta_2 u_k^2}{(1-\beta_2) P_{k-1} u_k^2 + 1} \quad (19)$$

ii) for any initial value $P_0 > 0$

$$\left(\frac{1}{P_0} + \beta_2 M_2^2 k\right)^{-1} \le P_k \le \left(\frac{1}{P_0} + \frac{\beta_2 M_1^2}{(1-\beta_2)P_0 M_2^2 + 1} k\right)^{-1}; \tag{20}$$

$$0 < P_{k+1} < P_k \; and \; \lim_{k\to\infty} P_k = 0. \tag{21}$$

Proof:

i) By (6) we have

$$1 - \frac{\beta_2 P_{k-1} u_k^2}{1 + P_{k-1} u_k^2} = \frac{P_k}{P_{k-1}}. \tag{22}$$

From $(1 + \beta_2 P_{k-1} u_k^2 / 1 + (1-\beta_2) P_{k-1} u_k^2)(1 - \beta_2 P_{k-1} u_k^2 / 1 + P_{k-1} u_k^2) = 1$, it follows $1 + \beta_2 P_{k-1} u_k^2 / 1 + (1-\beta_2) P_{k-1} u_k^2 = P_{k-1}/P_k$. Thus, (19) holds.

ii) For any initial value $P_0 > 0$, by $\beta_2 \in (0,1)$ and (22) we have $0 < P_{k+1} < P_k$. Noticing (11), we can get $\beta_2 M_1^2/(1-\beta_2)P_0 M_2^2 + 1 \le \beta_2 u_k^2/(1-\beta_2)P_{k-1} u_k^2 + 1 \le \beta_2 M_2^2$. This together with (19) implies (20). Furthermore, $\lim_{k\to\infty} P_k = 0$. □

Lemma 5: Under the conditions of Theorem 1, if $2\beta B > (M_2/M_1)^2$ and $E\tilde{\theta}_k^{2m} = o\left(k^{-1}\right)$ hold for some positive integer $m \ge 2$, then

$$E\tilde{\theta}_k^{2r} = o(k^{-1}), \quad 2 \le r \le m. \tag{23}$$

Proof: From (14), $|q_k| \le 1$ and

$$\frac{P_{k-1}|u_k|}{1 + P_{k-1}u_k^2} \le \frac{M_2}{M_1^2 + P_0^{-1} + \frac{\beta_2 M_1^2 (k-1)}{(1-\beta_2)P_0 M_2^2 + 1}} = O(k^{-1})$$

letting $\alpha = 2\bar{\theta} M_2$, by (9), (20) and Lemma 1, we have

$$\begin{aligned} E\tilde{\theta}_k^{2(m-1)} &\le E\tilde{\theta}_{k-1}^{2(m-1)} + o\left(\frac{1}{k^2}\right) \\ &\quad + E\left(\frac{2(m-1)\beta_1 P_{k-1} u_k}{1 + P_{k-1}u_k^2} \tilde{\theta}_{k-1}^{2(m-1)-1} \times \left(1 - 2\Phi(u_k \tilde{\theta}_{k-1})\right)\right) \\ &\le \left(1 - \frac{2(m-1)\beta B \left(\frac{M_1}{M_2}\right)^2}{k - 1 + \frac{(M_2^2 + P_0^{-1})}{(\beta_2 M_2^2)}}\right) \times E\tilde{\theta}_{k-1}^{2(m-1)} + o\left(\frac{1}{k^2}\right). \end{aligned}$$

Consequently, by Corollary 2, we can get $E\tilde{\theta}_k^{2(m-1)} = o\left(1/k\right)$. Similarly, by repeating this process, we can prove that (23) holds for all $r = 2, \ldots, m$. □

Lemma 6: Under the conditions of Theorem 1, if m is a positive integer satisfying $m > M_2^2/2\beta_1 \bar{B}_1 M_1^2$, then we have $E\tilde{\theta}_k^{2m} = o\left(k^{-1}\right)$.

Proof: Letting $\alpha = 2\bar{\theta} M_2$, by Lemma 1, there exists $\bar{B}_1 = \bar{B}_1(\alpha)$ such that

$$u_k \tilde{\theta}_{k-1}\left(1 - 2\Phi(u_k \tilde{\theta}_{k-1})\right) \le -\bar{B}_1 u_k^2 \tilde{\theta}_{k-1}^2.$$

Similar to the proof of Lemma 5, we have

$$\begin{aligned} E\tilde{\theta}_k^{2m} &\le E\tilde{\theta}_{k-1}^{2m} + o\left(k^{-2}\right) \\ &\quad + E\left(\frac{2m\beta_1 P_{k-1} u_k}{1 + P_{k-1}u_k^2} \tilde{\theta}_{k-1}^{2m-1} \times \left(1 - 2\Phi(u_k \tilde{\theta}_{k-1})\right)\right) \\ &\le \left(1 - \frac{2m\beta_1 \bar{B}_1 \left(\frac{M_1}{M_2}\right)^2}{k - 1 + \frac{(M_2^2 + P_0^{-1})}{(\beta_2 M_2^2)}}\right) E\tilde{\theta}_{k-1}^{m} \\ &\quad + o\left(\frac{1}{k^2}\right). \end{aligned}$$

This together with Corollary 2 implies $E\tilde{\theta}_k^{2m} = o(k^{-1})$. □

REFERENCES

[1] J. C. Aguero, G. C. Goodwin, and J. I. Yuz, "System identification using quantized data," in *Proc. 46th IEEE Conf. Decision Control*, 2007, pp. 4263–4268.

[2] M. Casini, A. Garulli, and A. Vicino, "Time complexity and input design in worst-case identification using binary sensors," in *Proc. 46th IEEE Conf. Decision Control*, New Orleans, LA, Dec. 12–14, 2007, pp. 5528–5533.

[3] L. Y. Wang, J. F. Zhang, and G. Yin, "System identification using binary sensors," *IEEE Trans. Autom. Control*, vol. 48, no. 11, pp. 1892–1907, Nov. 2003.

[4] L. Y. Wang, C. Y. Li, G. Yin, L. Guo, and C. Z. Xu, "State observability and observers of linear-time-invariant systems under irregular sampling and sensor limitations," *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2639–2654, Nov. 2011.

[5] K. You, L. H. Xie, S. Sun, and W. Xiao, "Multiple-level quantized innovation Kalman filter," in *Proc. 17th IFAC World Congress*, Jul. 6–11, 2008, pp. 1420–1425.

[6] A. Ribeiro, G. Giannakis, and S. Roumeliotis, "SOI-KF: Distributed Kalman filtering with low- cost communications using the sign of innovations," *IEEE Trans. Signal Processing*, vol. 54, no. 12, pp. 4782–4795, Dec. 2006.

[7] L. Y. Wang, Y. Kim, and J. Sun, "Prediction of oxygen storage capacity and stored NOx using HEGO sensor model for improved LNT control strategies," in *Proc. ASME Int. Mech. Eng. Congr. Expo.*, New Orleans, LA, Nov. 17-22, 2002, pp. 777–785.

[8] L. Y. Wang, G. Yin, J. F. Zhang, and Y. L. Zhao, *System Identification With Quantized Observations*. Boston, MA: Birkhauser, 2010.

[9] I. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, "Wireless sensor networks: A survey," *Computer Networks*, vol. 38, pp. 393–422, 2002.

[10] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints-Part I: State estimation problems," *IEEE Trans. Autom. Control*, vol. 42, no. 9, pp. 1294–1299, Sep. 1997.

[11] N. Dokuchaev and A. Savkin, "A new class of hybrid dynamical systems: State estimators with bit-rate constraints," *Int. J. Hybrid Syst.*, vol. 1, no. 1, pp. 33–50, 2001.

[12] G. Nair and R. Evans, "State estimation under bitrate constraints," in *Proc. 37th IEEE Conf. Decision Control*, Tampa, FL, 1998, pp. 251–256.

[13] L. Y. Wang and G. Yin, "Asymptotically efficient parameter estimation using quantized output observations," *Automatica*, vol. 43, pp. 1178–1191, 2007.

[14] Y. L. Zhao and J. Guo, "Quantization system identification under a class of persistent excitations," in *Proc. 29th Chinese Control Conf.*, Beijng, China, Jul. 29–31, 2010, pp. 6221–6226.

[15] L. Y. Wang, G. H. Xu, and G. Yin, "State reconstruction for linear time-invariant systems with binary-valued output observations," *Syst. Control Lett.*, vol. 57, pp. 958–963, 2008.

[16] L. Guo, "Convergence and logarithm laws of self-tuning regulators," *Automatica*, vol. 31, no. 3, pp. 435–450, 1995.

[17] L. Guo, *Time-Varying Stochastic Systems—Stability, Estimation and Control*. Changchun, China: Jilin Science and Technolory Press, 1993.

[18] H. F. Chen, *Stochastic Approximation and Its Application*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2002.

[19] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed. New York: Springer-Verlag, 1997.